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The q-deformed Poisson bracket, Levi-Civita symbol and Poincaré algebra

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Abstract. In this paper we use a special choice of the $GL_{X,q_{ij}}(2N)$ quantum plane and its differential calculus for a q-deformed phase space to define a modified Poisson bracket and construct the contraction rule for the q-deformed Levi-Civita symbol. We find the q-deformed phase-space variable realization of the $so_q(3)$ and q-deformed Poincaré algebras.

1. Introduction

It is known that the quantum Yang–Baxter equation plays a crucial role in diverse problems in theoretical physics. These include exactly soluble models in statistical physics [1] and quantum integrable model field theory [2–9]. Quantum groups provide a practical example of a non-commutative differential geometry [10]. The idea of a quantum plane was first introduced by Manin [11–13]. The non-commutative differential geometry was first applied to quantum matrix groups by Woronowicz [14, 15]. However, it is Wess and Zumino [16, 17] who considered one of the simplest examples of non-commutative differential calculus on Manin's quantum plane. They developed a differential calculus on the quantum hyperplane which was covariant with respect to the action of the quantum deformation of $GL(n)$, the so called $GL_q(n)$. After this, much work followed in this direction [18–26]. In spite of this, it is still uncertain whether this new mathematical object will, in future, bring new 'phenomena' into physics or not. Since the symmetries play an important role in physics, it is worth extending them to the deformed concept of symmetries which might also be used in physics. If quantum groups are applied to some types of physics, they are supposed to create a type of 'new' physics which defaults back to its classical version when the deformation parameters take particular values. To this end it is worthwhile constructing the fundamental concepts of and computational techniques for quantum groups.

Recently some papers have described the q-deformed Poincaré algebra [27–31]. This paper should be included among them. However, it differs from them in that it starts from the q-deformed Poisson bracket. Therefore we can say that the context of this paper is an example of the q-deformation of classical theory, not quantum theory.

In this paper we make a special choice for the q-deformed phase space and differential calculus. We also construct the contraction rule for the q-deformed Levi-Civita symbol and prove it. We use these results to obtain a classical q-deformed $so(3)$ algebra and a classical q-deformed Poincaré algebra.

2. The q-phase space and q-Poisson bracket

In this section we introduce a special choice for the q-deformed phase space to define the q-deformed Poisson bracket. First, let us define the local variables $(x_i, p_i, i = 1, 2, \ldots, N)$

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of the q-deformed phase space so as to satisfy the following commutation relation

$$\begin{aligned} &x_i p_i = p_i x_i \qquad x_i x_j = q^{-1} x_j x_i \\ &p_i p_j = q^{-1} p_j p_i \qquad x_i p_j = q p_j x_i \\ &p_i x_j = q x_j p_i \qquad (i < j, i, j = 1, 2, \ldots, N). \end{aligned} \tag{1}$$

From these relations we conclude that each pair of the q-deformed phase-space variables (x_i, p_i) for every $i = 1, 2, \ldots, N$ describes the ordinary plane where x_i and p_i are mutually commuting. However, the interconnection of different planes is described by q-deformed space relations. We call this q-deformed space the q-deformed Poisson manifold.

In order to define the q-deformed Poisson bracket, it is necessary to consider the q-classical observables which are functions of q-classical phase-space variables $x_i, p_i, (i = 1, 2, \ldots, N)$. Let $f(X, P)$ be a monomial whose form is

$$f(X, P) = x_1^{m_1} p_1^{n_1} x_2^{m_2} p_2^{n_2} \ldots x_N^{m_N} p_N^{n_N} \tag{2}$$

where X and P denote $(x_1, \ldots, x_N)$ and $(p_1, \ldots, p_N)$, respectively. From now on we will say that $x_1^{m_1} p_1^{n_1} x_2^{m_2} p_2^{n_2} \ldots x_N^{m_N} p_N^{n_N}$ belongs to the $(M_1, M_2, \ldots, M_N)$-class where

$$M_1 = m_1 - n_1 \qquad M_2 = m_2 - n_2, \ldots \qquad M_N = m_N - n_N.$$

At this stage we define the q-Poisson bracket for two monomials f and g as follows:

$$\begin{aligned} \{f, g\}_q &= \sum_{i=1}^{N} \left[q^{\sum_{k=1}^{i-1} M_k - \sum_{k=i+1}^{N} M_k} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p_i} - q^{-\sum_{k=1}^{i-1} M_k + \sum_{k=i+1}^{N} M_k} \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x_i} \right] \\ &= \sum_{i=1}^{N} \left[f \overleftarrow{\frac{\partial}{\partial x_i}} \overrightarrow{\frac{\partial}{\partial p_i}} g - f \overleftarrow{\frac{\partial}{\partial p_i}} \overrightarrow{\frac{\partial}{\partial x_i}} g \right] \end{aligned} \tag{3}$$

where the left derivatives $\overleftarrow{\partial/\partial x_i}$ and $\overleftarrow{\partial/\partial p_i}$ act on $f(X, P)$ from the left, and the right derivatives $\overrightarrow{\partial/\partial x_i}$ and $\overrightarrow{\partial/\partial p_i}$ act on $g(X, P)$ from the right. The relation between the right and left derivatives is

$$q^{\sum_{k=1}^{i-1} M_k - \sum_{k=i+1}^{N} M_k} \overrightarrow{\frac{\partial}{\partial x_i}} f = f \overleftarrow{\frac{\partial}{\partial x_i}} \tag{4}$$

$$q^{-\sum_{k=1}^{i-1} M_k + \sum_{k=i+1}^{N} M_k} \overrightarrow{\frac{\partial}{\partial p_i}} f = f \overleftarrow{\frac{\partial}{\partial p_i}}. \tag{5}$$

For future calculations we propose that the q-deformed Poisson bracket fulfils

$$\{af_1 + bf_2, g\}_q = a\{f_1, g\}_q + b\{f_2, g\}_q$$

where a and b are real fields and monomials f_1, f_2 and g belongs to different (or the same) class. The relations are obtained by using the commutation relations between the q-phase space variables and their derivatives:

$$\begin{aligned} &\frac{\partial}{\partial x_j} p_i = q p_i \frac{\partial}{\partial x_j} \qquad \frac{\partial}{\partial p_j} x_i = q x_i \frac{\partial}{\partial p_j} \\ &\frac{\partial}{\partial x_j} x_i = q^{-1} x_i \frac{\partial}{\partial x_j} \qquad \frac{\partial}{\partial p_j} p_i = q^{-1} p_i \frac{\partial}{\partial p_j} \qquad (i < j). \end{aligned} \tag{6}$$

Equation (6) for $i > j$ can be obtained by replacing q with q^{-1}. The proof of equation (6) is easy. To start with we prove the first relation of (6). Using the left-hand side of the first relation of (6) on a monomial $\Pi_{k=1}^{N} x_k^{m_k} \Pi_{k=1}^{N} p_k^{n_k}$ leads to

$$\begin{aligned}
\frac{\partial}{\partial x_j} p_i \prod_{k=1}^{N} x_k^{m_k} \prod_{k=1}^{N} p_k^{n_k} &= \frac{\partial}{\partial x_j} p_i x_j^{m_j} \prod_{k=1,k\neq j}^{N} x_k^{m_k} \prod_{k=1}^{N} p_k^{n_k} \\
&= \frac{\partial}{\partial x_j} q^{m_j} x_j^{m_j} p_i \prod_{k=1,k\neq j}^{N} x_k^{m_k} \prod_{k=1}^{N} p_k^{n_k} \\
&= q^{m_j} m_j x_j^{m_j-1} p_i \prod_{k=1,k\neq j}^{N} x_k^{m_k} \prod_{k=1}^{N} p_k^{n_k} \\
&= q^{m_j} q^{-(m_j-1)} m_j p_i x_j^{m_j-1} \prod_{k=1,k\neq j}^{N} x_k^{m_k} \prod_{k=1}^{N} p_k^{n_k} \\
&= q p_i \frac{\partial}{\partial x_j} \prod_{k=1}^{N} x_k^{m_k} \prod_{k=1}^{N} p_k^{n_k}.
\end{aligned}$$

The other relations of (6) can be obtained in a similar manner.

If $f(X, P)$ and $g(X, P)$ belong to the $(M_1, M_2, \ldots, M_N)$-class and $(L_1, L_2, \ldots, L_N)$-class, respectively, then $f(X, P)g(X, P)$ belongs to the $(M_1+L_1, M_2+L_2, \ldots, M_N+L_N)$-class. Since $(\partial/\partial x)x_1 = 1$ and 1 and x_1 belong to the $(0, 0, \ldots, 0)$-class and $(1, 0, \ldots, 0)$-class, respectively, $\partial/\partial x_1$ belongs to the $(-1, 0, \ldots, 0)$-class. Similarly we see that $\partial/\partial p_1$ belongs to the $(1, 0, \ldots, 0)$-class, etc. Then the commutation relation between $f(X, P)$ and $g(X, P)$ is given by

$$f(X, P)g(X, P) = q^{\sum_{i=1}^{N} M_i\left(-\sum_{j=1}^{i-1} L_j + \sum_{j=i+1}^{N} L_j\right)} g(X, P) f(X, P). \tag{7}$$

From the above definition we can easily see that two elements belonging to the same class commute. From the commutation relation between two elements belonging to their respective distinct class, we obtain

$$\{f, g\}_q = -q^{\sum_{i=1}^{N} M_i\left(-\sum_{j=1}^{i-1} L_j + \sum_{j=i+1}^{N} L_j\right)} \{g, f\}_q \tag{8}$$

where $f(X, P)$ and $g(X, P)$ belong to the $(M_1, M_2, \ldots, M_N)$-class and $(L_1, L_2, \ldots, L_N)$-class, respectively.

Therefore, if $f(X, P)$ and $g(X, P)$ belong to the same class, we have

$$\{f, g\}_q = -\{g, f\}_q. \tag{9}$$

It is worth noting that the arbitrary q-classical observables consist of several elements belonging to their respective distinct classes. The q-Jacobi identity is written as

$$\begin{aligned}
&\{f, \{g, h\}_q\}_q + q^{\sum_{i=1}^{N}(M_i+L_i)\left(-\sum_{j=1}^{i-1} R_j + \sum_{j=i+1}^{N} R_j\right)} \{h, \{f, g\}_q\}_q \\
&\quad + q^{\sum_{i=1}^{N} M_i\left(-\sum_{j=1}^{i-1}(L_j+R_j) + \sum_{j=i+1}^{N}(L_j+R_j)\right)} \{g, \{h, f\}_q\}_q = 0
\end{aligned} \tag{10}$$

where monomials f, g and h are assumed to belong to the $(M_1, \ldots, M_N)$-class, the $(L_1, \ldots, L_N)$-class and the $(R_1, \ldots, R_N)$-class, respectively.

3. Contraction rule for the q-deformed Levi-Civita symbol

In this section we obtain a q-analogue of the contraction rule for the q-deformed Levi-Civita symbol, which is defined as

$$E_{12\ldots n} = 1 \tag{11}$$

and

$$E_{\ldots ij\ldots} = (-q)^{P(i,j)} E_{\ldots ji\ldots} \tag{12}$$

where $P(i, j)$ is defined as

$$P(i,j) = 1 \qquad (i > j)$$
$$P(i,j) = -1 \qquad (i < j).$$

For example, the q-Levi-Civita symbol of rank three is easily computed according to definitions (11) and (12);

$$E_{123} = 1$$
$$E_{132} = (-q)E_{123} = -q$$
$$E_{213} = (-q)E_{123} = -q$$
$$E_{231} = (-q)E_{213} = (-q)^2 E_{123} = (-q)^2$$
$$E_{312} = (-q)E_{132} = (-q)^2 E_{123} = (-q)^2$$
$$E_{321} = (-q)E_{231} = (-q)^2 E_{213} = (-q)^3 E_{123} = (-q)^3.$$

When q goes to 1, the above equations reduce to 1 (or -1) for even (or odd) permutation of $(1, 2, 3)$.

To begin with we write down the q-deformed contraction rule for the q-deformed Levi-Civita symbol and prove it later:

$$E_{i_1\ldots i_N k} E_{j_1\ldots j_N k} = q^{2\left(\sum_{l=1}^{N} i_l - S(i_1,\ldots,i_N) - N\right)} \sum_{\pi \in S_N} E^{j_1\ldots j_N k}_{i_1\ldots i_N k} \delta^{i_1}_{\pi(j_1)} \delta^{i_2}_{\pi(j_2)} \cdots \delta^{i_N}_{\pi(j_N)} \tag{13}$$

where S_N means the permutation group of degree N and

$$E^{j_1\ldots j_N k}_{i_1\ldots i_N k} = \frac{E_{i_1\ldots i_N}}{E_{j_1\ldots j_N}}. \tag{14}$$

Here $S(i_1, \ldots, i_N)$ is defined as

$$S(i_1, \ldots, i_N) = \sum_{n=1}^{N-1} \sum_{m=n+1}^{N} S(i_n, i_m)$$

where

$$S(i,j)=1 \qquad (\text{if } i<j)$$
$$S(i,j)=0 \qquad (\text{if } i\geqslant j).$$

For example, $S(1,3,2,4)$ is computed as follows:

$$S(1,3,2,4)=S(1,3)+S(1,2)+S(1,4)+S(3,2)+S(3,4)+S(2,4)=5.$$

Now we will prove the property of q-Levi-Civita symbol (13) by means of mathematical induction.

Let us assume that equation (13) holds for q-Levi-Civita symbol of rank N. First we can easily obtain an equivalent form of equation (13) as follows:

$$E_{i_1\dots i_N k}E_{j_1\dots j_N k}=q^{2\left(\sum_{l=1}^{N} i_l-S(i_1,\dots,i_N)-N\right)}\sum_{l=1}^{N}\delta_{j_l}^{i_1}\sum_{\pi_l\in S_{N-1}(\hat{j}_l)}E_{i_1\dots i_N k}^{j_1\dots j_N k}\delta_{\pi_l(j_1)}^{i_2}\delta_{\pi_l(j_2)}^{i_3}\dots\delta_{\pi_l(j_N)}^{i_N} \tag{17}$$

where $S_{N-1}(\hat{j}_l)$ means the permutation group of degree $N-1$ where j_l is deleted.

Consider the case $i_1=j_l=I,\ I=1,2,\dots,N$. From the definition of the q-Levi-Civita symbol we obtain

$$E_{i_1\dots i_N k}E_{j_1\dots j_N k}=(-q)^{2(I-1)+\sum_{k=1}^{l-1}P(j_k,j_l)}E_{i_2\dots i_N k}E_{j_1\dots\hat{j}_l\dots j_N k}. \tag{18}$$

Since we have assumed that the q-contraction rule for the q-Levi-Civita symbol holds for the rank-N case, we have

$$\begin{aligned}E_{i_1\dots i_N k}E_{j_1\dots j_N k}&=(-q)^{2(I-1)+\sum_{k=1}^{l-1}P(j_k,j_l)}q^{2\left(\sum_{l=2}^{N} i_l+I-N-S(i_2,\dots,i_N)-(N-1)\right)}\\&\quad\times\sum_{\pi_l\in S_{N-1}(\hat{j}_l)}E_{i_2\dots\dots\dots i_N k}^{j_1\dots\hat{j}_l\dots j_N k}\delta_{\pi_l(j_1)}^{i_2}\delta_{\pi_l(j_2)}^{i_3}\dots\delta_{\pi_l(j_N)}^{i_N}.\end{aligned} \tag{19}$$

Using the relation

$$E_{i_2\dots\dots\dots i_N k}^{j_1\dots\hat{j}_l\dots j_N k}=(-q)^{-\sum_{k=1}^{l-1}P(j_k,j_l)}E_{i_1\dots i_N k}^{j_1\dots j_N k}\qquad \text{for } i_1=j_l=I \tag{20}$$

we have

$$E_{i_1\dots i_N k}E_{j_1\dots j_N k}=q^{2\left(\sum_{l=1}^{N} i_l-(S(i_2,\dots,i_N)+N-I)-N\right)}\sum_{\pi_l\in S_{N-1}(\hat{j}_l)}E_{i_1\dots i_N k}^{j_1\dots j_N k}\delta_{\pi_l(j_1)}^{i_2}\delta_{\pi_l(j_2)}^{i_3}\dots\delta_{\pi_l(j_N)}^{i_N}. \tag{21}$$

From the definition of $S(i_1,\dots,i_N)$, we have for $i_1=j_l=I$

$$\begin{aligned}S(I,i_2,\dots,i_N)&=\sum_{m=2}^{N}S(I,i_m)+S(i_2,\dots,i_N)\\&=N-I+S(i_2,\dots,i_N).\end{aligned} \tag{22}$$

Inserting equation (22) into the right-hand side of equation (21), we complete the proof of equation (13) by virtue of the induction principle.

Equation (13) can be generalized into a more generic form, which is written as

$$E_{i_1 i_2 \ldots i_N} E_{j_1 j_2 \ldots j_N} \delta_{i_m j_l} = (-q)^{2\left[\sum_{k=1}^{m-1} i_k - \sum_{k=m+1}^{N} i_k + S'(i_1, \ldots, \hat{i_m}, \ldots, i_N) + (N-m)(N-m+1) - (m-1)\right]} \times \sum_{\pi \in S_{N-1}} E^{j_1 \ldots j_N}_{i_1 \ldots i_N} \delta^{i_1 \ldots \hat{i_m} \ldots i_N}_{\pi(j_1) \ldots \pi(\hat{j_l}) \ldots \pi(j_N)} \tag{23}$$

where

$$E^{j_1 \ldots j_N}_{i_1 \ldots i_N} = E_{j_1 \ldots j_N} / E_{i_1 \ldots i_N}$$

$$S'(i_1, \ldots, \hat{i_m}, \ldots, i_N) = \sum_{l=1}^{m-1} \sum_{n=l+1}^{m-1} S(i_l, i_n) + \sum_{l=m+1}^{N} \sum_{n=l+1}^{N} S(i_l, i_n) - \sum_{l=1}^{m-1} \sum_{n=m+1}^{N} S(i_l, i_n).$$

The formulae for $N = 3$ and $N = 4$ are listed in appendices A and B.

4. Classical q-deformed $so(3)$ algebra

In this section we use the q-deformed Poisson bracket for $N = 3$ to construct the phase-space variable realization of the classical q-deformed $su(3)$ algebra. Throughout, we will write the classical q-deformed $su(3)$ algebra as the $su_q(3)$ algebra, but this algebra does not mean the ordinary $su_q(3)$ algebra at the quantum level. Now we assume that the three generators of $su_q(3)$ take a form similar to that in the non-deformed case:

$$L_i = E_{ijk} x_j p_k. \tag{24}$$

The concrete form of the three generators are given by

$$\begin{aligned} L_1 &= x_2 p_3 - q x_3 p_2 \\ L_2 &= q^2 x_3 p_1 - q x_1 p_3 \\ L_3 &= q^2 x_1 p_2 - q^3 x_2 p_1 \end{aligned} \tag{25}$$

where L_1, L_2 and L_3 are three generators of $su_q(3)$. Here L_1 consists of the element belonging to the $(0, 1, -1)$-class and that belonging to the $(0, -1, 1)$-class, L_2 consists of the element belonging to the $(-1, 0, 1)$-class and the element belonging to the $(1, 0, -1)$-class and L_3 consists of the element belonging to the $(1, -1, 0)$-class and the element belonging to the $(-1, 1, 0)$-class. Here E_{ijk} is called the q-Levi-Civita symbol and its non-vanishing components are

$$\begin{aligned} &E_{123} = 1 \qquad &&E_{132} = -q \qquad &&E_{213} = -q \\ &E_{231} = (-q)^2 \qquad &&E_{312} = (-q)^2 \qquad &&E_{321} = (-q)^3. \end{aligned} \tag{26}$$

The $su_q(3)$ algebra is written as

$$\{L_i, L_l\}_q = (\theta_{li} q^{2l-5} + \theta_{il} q^{2l-3}) E_{ilm} L_m \tag{27}$$

where

$$\theta_{li} = 1 \qquad (l > i)$$

$$\theta_{li} = 0 \qquad (l \leqslant i).$$

The proof of equation (27) is as follows:

$$\begin{aligned}\{L_i, L_l\}_q &= E_{ijk}E_{lmn}\{x_j p_k, x_m p_n\}_q \\ &= E_{ijk}E_{lmn}\left[x_j p_k \frac{\overleftarrow{\partial}}{\partial x_r}\frac{\overrightarrow{\partial}}{\partial p_r} x_m p_n - x_j p_k \frac{\overleftarrow{\partial}}{\partial p_r}\frac{\overrightarrow{\partial}}{\partial x_r} x_m p_n\right]\end{aligned}$$

where

$$x_j p_k = q^{-P(j,k)} p_k x_j.$$

Using relations (4), (5) we have

$$\begin{aligned}\{L_i, L_l\}_q &= E_{ijk}E_{lmn}[q^{-P(j,k)-P(m,n)} p_k x_m \delta_{jr}\delta_{nr} - x_j p_n \delta_{kr}\delta_{mr}] \\ &= E_{ikj}E_{ljm} p_k x_m - E_{ijk}E_{lkn} x_j p_n \\ &= E_{ijk}E_{lkn}(p_j x_n - x_j p_n) \\ &= -\theta_{ji} q^{2i-1}(\delta^i_l \delta^j_n - q^{2(j-i)-1}\delta^i_n \delta^j_l)(p_j x_n - x_j p_n) \\ &\quad - \theta_{ij} q^{2i-1}(\delta^i_l \delta^j_n - q^{2(j-i)+1}\delta^i_n \delta^j_l)(p_j x_n - x_j p_n) \\ &= \theta_{ji} q^{2j-2}(\delta^j_l p_j x_i - \delta^j_l x_j p_i) + \theta_{ij} q^{2j}(\delta^j_l p_j x_i - \delta^j_l x_j p_i) \\ &= \theta_{li} q^{2l-3}(x_i p_l - q x_l p_i) + \theta_{il} q^{2l+1}(x_i p_l - q^{-1} x_l p_i) \\ &= (\theta_{li} q^{2l-5} + \theta_{il} q^{2l-3}) E_{ilm} L_m\end{aligned}$$

which completes the proof of equation (27).

Now we should demonstrate that the algebra (27) is really a deformation, i.e. that the q-factors cannot be eliminated by rescaling the generators. The algebra can be rewritten as

$$\{L_1, L_2\}_q = q^{-1} L_3 \qquad \{L_2, L_3\}_q = q^3 L_1 \qquad \{L_1, L_3\}_q = -q^2 L_2$$

$$\{L_2, L_1\}_q = -L_3 \qquad \{L_3, L_1\}_q = q L_2 \qquad \{L_3, L_2\}_q = -q^4 L_1.$$

If we rescale $L_i \to k_i L_i$ to eliminate the q-factors, the first and fourth relations give

$$k_1 k_2 = q^{-1} k_3 \qquad k_2 k_1 = k_3$$

which gives $q^{-1} = 1$ for non-vanishing k_i. This implies that the algebra (27) is really a deformation and that the q-factors cannot be eliminated.

Now we will show that the algebra (27) is homomorphic to Fairlie's Cartesian $su_q(2)$ algebra [32] by rescaling the generators. First we can see that the algebra (27) is homomorphic to the following commutation relations

$$[L_1, L_2]_{q^{-1}} = q^{-1} L_3 \qquad [L_2, L_3]_{q^{-1}} = q^3 L_1 \qquad [L_3, L_1]_{q^{-1}} = q L_2$$

$$[L_2, L_1]_q = -L_3 \qquad [L_3, L_2]_q = -q^4 L_1 \qquad [L_1, L_3]_q = -q^2 L_2$$

where commutator $[A, B]_q$ is defined as

$$[A, B]_q = AB - qBA.$$

By rescaling the generators

$$L_i \to k_i L_i \qquad i = 1, 2, 3$$

we have

$$q\frac{k_1k_2}{k_3}L_1L_2 - \frac{k_1k_2}{k_3}L_2L_1 = L_3$$

$$q^{-1}\frac{k_3k_1}{k_2}L_3L_1 - q^{-2}\frac{k_3k_1}{k_2}L_1L_3 = L_2$$

$$q^{-3}\frac{k_2k_3}{k_1}L_2L_3 - q^{-4}\frac{k_2k_3}{k_1}L_3L_2 = L_1.$$

Comparing the above relations with the Fairlie algebra

$$rXY - r^{-1}YX = Z$$

$$rYZ - r^{-1}ZY = X$$

$$rZX - r^{-1}XZ = Y$$

we should demand that

$$q\frac{k_1k_2}{k_3} = q^{-1}\frac{k_3k_1}{k_2} = q^{-3}\frac{k_2k_3}{k_1} = r$$

$$\frac{k_1k_2}{k_3} = q^{-2}\frac{k_3k_1}{k_2} = q^{-4}\frac{k_2k_3}{k_1} = r^{-1}.$$

Solving the above relation we have

$$k_1 = q^{1/2} \qquad k_2 = q^{3/2} \qquad k_3 = q^{5/2} \qquad \text{and} \qquad r = q^{1/2}.$$

Therefore the algebra (27) is homomorphic to the following algebra

$$q^{1/2}L_1L_2 - q^{-1/2}L_2L_1 = L_3$$

$$q^{1/2}L_2L_3 - q^{-1/2}L_3L_2 = L_1$$

$$q^{1/2}L_3L_1 - q^{-1/2}L_1L_3 = L_2$$

which is consistent with Fairlie's Cartesian $su_q(2)$ algebra given in [32]. We can construct the q-deformed harmonic Hamiltonian in three dimensions and we show that H is in involution, that is to say,

$$\{H, H\}_q = 0 \tag{28}$$

where the q-deformed harmonic Hamiltonian is given by

$$H = \sum_{i=1}^{3}(p_i^2 + x_i^2). \tag{29}$$

Here equation (28) results from the fact that x_i and p_i for every $i = 1, 2, \ldots, N$ commutes.

5. Classical q-deformed Poincaré algebra

In this section we obtain the q-analogue of the Poincaré algebra by means of the q-deformed Poisson bracket. In analogy with equation (24), we introduce the ten independent generators of the q-deformed Poincaré algebra. The four generators describe the q-deformed translation:

$$P_i = p_i \qquad i = 1, 2, 3, 4. \tag{30}$$

The six other generators describe the q-deformed Lorentz generators:

$$M_{ij} = E_{ijkl} x_k p_l \qquad (i < j). \tag{31}$$

The concrete forms of the six q-deformed Lorentz generators are written as

$$\begin{aligned}
M_{12} &= x_3 p_4 - q x_4 p_3 \\
M_{13} &= -q x_2 p_4 + q^2 x_4 p_2 \\
M_{23} &= q^2 x_1 p_4 - q^3 x_4 p_1 \\
M_{14} &= q^2 x_2 p_3 - q^3 x_3 p_2 \\
M_{24} &= -q^3 x_1 p_3 + q^4 x_3 p_1 \\
M_{34} &= q^4 x_1 p_2 - q^5 x_2 p_1.
\end{aligned} \tag{32}$$

The first three generators of equation (32) mean a q-deformed boost for each direction and the second three generators imply a q-deformed space rotation which is given in section 3. At this stage we cannot but confess that we do not know the 'physical' meaning of the q-deformed boost, the q-deformed space rotation or the q-deformed translation.

Then the q-deformed Poincaré algebra is written as follows

$$\{P_i, P_j\}_q = 0, \tag{33}$$

$$\{M_{ij}, P_k\}_q = q^{-P(k,l)} E_{ijkl} P_l \tag{34}$$

$$\begin{aligned}
\{M_{ij}, M_{kl}\}_q = &-\delta_{ik}[\theta_{lj}(-q)^{2l+2i-12} + \theta_{jl}(-q)^{(3-i)l+5i-11}] E_{ljab} M_{ab} \\
&- \delta_{jl}[\theta_{ki}(-q)^{(j-2)k-4} + \theta_{ik}(-q)^{2k+2j-8}] E_{kiab} M_{ab} \\
&- \delta_{jk}(-q)^{6j+(4-j)l-21} E_{liab} M_{ab} \\
&- \delta_{il}[\theta_{kj}(-q)^{(4-i)k+6i-21} + \theta_{jk}(-q)^{(i-1)k+i-4}] E_{kjab} M_{ab}
\end{aligned} \tag{35}$$

where the repeated indices a, b are assumed to be summed over $a < b$. The proof of equation (35) is as follows

$$\begin{aligned}
\{M_{ij}, M_{kl}\}_q &= E_{ijmn} E_{klpq} \{x_m p_n, x_p p_q\}_q \\
&= E_{ijmn} E_{klpq} \left(x_m p_n \frac{\overleftarrow{\partial}}{\partial x_r} \frac{\overrightarrow{\partial}}{\partial p_r} x_p p_q - x_m p_n \frac{\overleftarrow{\partial}}{\partial p_r} \frac{\overrightarrow{\partial}}{\partial x_r} x_p p_q \right) \\
&= E_{ijmn} E_{klpq} (q^{-P(p,q)-P(m,n)} p_n x_p \delta_{mr} \delta_{qr} - x_m p_q \delta_{nr} \delta_{pr}) \\
&= E_{ijmp} E_{klpq} (p_m x_q - x_m p_q).
\end{aligned}$$

First, consider the case that $j=k$ in $E_{ijmp}E_{klpq}$, then the contraction rule is given by

$$E_{ijmp}E_{klpq}=(-q)^{(j-1)(i+1)}(\delta^i_l\delta^m_q+(-q)^{(4-j)m-(j-1)i+5j-14}\delta^i_q\delta^m_l).$$

Then we have

$$\begin{aligned}\{M_{ij},M_{kl}\}_q&=(-q)^{6j-15+(4-j)l}(p_lx_i-x_lp_i)\qquad(\text{for } j=k)\\&=-(-q)^{6j-21+(4-j)l}E_{liab}M_{ab}\end{aligned}$$

where

$$E_{liab}M_{ab}=E_{liab}E_{abmn}x_mp_n=(-q)^5(x_ip_l-qx_lp_i).$$

For $i=k$ in $E_{ijmp}E_{klpq}$, we have

$$\begin{aligned}E_{ijmp}E_{klpq}&=\theta_{mj}(-q)^{(3-i)j+5i-8}(\delta^i_l\delta^m_q+(-q)^{2m-(3-i)j-3i+2}\delta^i_q\delta^m_l)\\&\quad+\theta_{jm}(-q)^{2j+2i-5}(\delta^i_l\delta^m_q+(-q)^{-2j+(3-i)m+3i-2}\delta^i_q\delta^m_l).\end{aligned}$$

Then we have

$$\begin{aligned}\{M_{ij},M_{kl}\}_q&=\theta_{lj}(-q)^{2l+2i-6}(p_lx_j-x_lP_j)+\theta_{jl}(-q)^{(3-i)l+5i-7}(p_lx_j-x_lP_j)\qquad(\text{for } i=k)\\&=[-\theta_{lj}(-q)^{2l+2i-12}-\theta_{jl}(-q)^{(3-i)l+5i-11}]E_{ljab}M_{ab}\end{aligned}$$

where we used the relation

$$E_{ljab}M_{ab}=E_{ljab}E_{abmn}x_mp_n=\theta_{lj}(-q)^5(x_jp_l-qx_lp_j)+\theta_{jl}(-q)^4(x_lp_j-qx_jp_l).$$

For $i=l$ in $E_{ijmp}E_{klpq}$, we have

$$\begin{aligned}E_{ijmp}E_{klpq}&=\theta_{mj}(-q)^{(i-1)j+i-1}(\delta^i_l\delta^m_q+(-q)^{-(i-1)j+(4-i)m+5i-14}\delta^i_q\delta^m_l)\\&\quad+\theta_{jm}(-q)^{(4-i)j+6i-14}(\delta^i_l\delta^m_q+(-q)^{-(4-i)j+(i-1)m+14-5i}\delta^i_q\delta^m_l).\end{aligned}$$

Then we have

$$\begin{aligned}\{M_{ij},M_{kl}\}_q&=\theta_{kj}(-q)^{(4-i)k+6i-15}(p_kx_j-x_kp_j)\\&\quad+\theta_{jk}(-q)^{(i-1)k+i}(p_kx_j-x_kp_j)\qquad(\text{for } i=l)\\&=-\theta_{kj}(-q)^{(4-i)k+6l-21}E_{kjab}M_{ab}-\theta_{jk}(-q)^{(i-1)k+i-4}E_{kjab}M_{ab}\end{aligned}$$

where we used

$$E_{kjab}M_{ab}=E_{kjab}E_{abmn}x_mp_n=\theta_{kj}(-q)^5(x_jp_k-qx_kp_j)+\theta_{jk}(-q)^4(x_kp_j-qx_jp_k).$$

Finally, for $j=l$ in $E_{ijmp}E_{klpq}$, we have

$$\begin{aligned}E_{ijmp}E_{klpq}&=\theta_{mi}(-q)^{2i+2j-5}(\delta^i_l\delta^m_q+(-q)^{-2i+(j-2)m+7-2j}\delta^i_q\delta^m_l)\\&\quad+\theta_{im}(-q)^{(4-j)i+3}(\delta^i_l\delta^m_q+(-q)^{2m-(4-j)i+2j-7}\delta^i_q\delta^m_l).\end{aligned}$$

Then we have

$$\{M_{ij}, M_{kl}\}_q = \theta_{ki}(-q)^{(j-2)k+2}(p_k x_i - x_k p_i) + \theta_{ik}(-q)^{2k+2j-4}(p_k x_i - x_k p_i) \qquad (\text{for } j = l)$$
$$= -\theta_{ki}(-q)^{(j-2)k-4} E_{kiab} M_{ab} - \theta_{ik}(-q)^{2k+2j-8} E_{kiab} M_{ab}$$

where we used

$$E_{kiab} M_{ab} = E_{kiab} E_{abmn} x_m p_n = \theta_{ki}(-q)^5 (x_i p_k - q x_k p_i) + \theta_{ik}(-q)^4 (x_k p_i - q x_i p_k).$$

Owing to the reason mentioned in section 4, we can see that the relations (33)–(35) are really deformations. We can easily show that the classical q-deformed Poincaré algebra (33)–(35) is homomorphic to some commutation relations in a way similar to that explained in section 4. However, Fairlie's construction for q-deformed Poincaré algebra is not yet known, so we cannot compare the algebra (33)–(35) with a known algebra. Considering that the $su_q(3)$ algebra given in equation (27) is homomorphic to Fairlie's Cartesian $su_q(2)$ algebra, we can guess that the classical q-deformed Poincaré algebra (33)–(35) may be related to the q-deformed Cartesian Poincaré algebra and that a deforming map into the ordinary Poincaré algebra may exist.

6. Conclusion

In this paper we define the q-Poisson bracket and use this to construct the $su_q(3)$ algebra and q-deformed Poincaré algebra in terms of q phase-space variables. There remains much work to be done in this direction.

First, this procedure should be quantized in a consistent manner. In doing so, we must construct the q-bracket correctly and investigate the relation between the q-Poisson bracket and q-bracket.

Second, the $su_q(3)$ algebra is deeply connected with the rotation in three dimensions. The question then is: which phenomena are connected with $su_q(3)$?

Third, we think this procedure should be extended to the $su_q(n)$ case. The Maxwell equation possesses $su(4)$ symmetry. Is there a q-analogue of the Maxwell equation and special relativity with symmetry group $su_q(4)$?

Finally, there is the question as to whether the classical q-deformed Poincaré algebra (33)–(35) is homomorphic to the Cartesian q-deformed Poincaré algebra.

We hope that these problems and their interesting related topics will be studied in the near future.

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Appendix A

$$E_{ijk}E_{lmk} = \theta_{ji}(-q)^{2(i+j-3)}(\delta^i_l\delta^j_m - q\delta^i_m\delta^j_l) + \theta_{ij}(-q)^{2(i+j-2)}(\delta^i_l\delta^j_m - q^{-1}\delta^i_m\delta^j_l) \quad (A.1)$$

$$E_{ijk}E_{lkm} = \theta_{ji}(-q)^{2i-1}(\delta^i_l\delta^j_m - q^{2(j-i)-1}\delta^i_m\delta^j_l) + \theta_{ij}(-q)^{2i-1}(\delta^i_l\delta^j_m - q^{2(j-i)+1}\delta^i_m\delta^j_l) \quad (A.2)$$

$$E_{ijk}E_{klm} = \theta_{ji}(-q)^{2}(\delta^i_l\delta^j_m - q\delta^i_m\delta^j_l) + \theta_{ij}(-q)^{4}(\delta^i_l\delta^j_m - q^{-1}\delta^i_m\delta^j_l) \quad (A.3)$$

$$E_{ikj}E_{lkm} = \theta_{ji}(-q)^{2(2+i-j)}(\delta^i_l\delta^j_m - q^{2(j-i)-1}\delta^i_m\delta^j_l) + \theta_{ij}(-q)^{2(1+i-j)}(\delta^i_l\delta^j_m - q^{2(j-i)+1}\delta^i_m\delta^j_l) \quad (A.4)$$

$$E_{ikj}E_{klm} = \theta_{ji}(-q)^{-2j+7}(\delta^i_l\delta^j_m - q\delta^i_m\delta^j_l) + \theta_{ij}(-q)^{-2j+7}(\delta^i_l\delta^j_m - q^{-1}\delta^i_m\delta^j_l) \quad (A.5)$$

$$E_{kij}E_{klm} = \theta_{ji}(-q)^{2(5-i-j)}(\delta^i_l\delta^j_m - q\delta^i_m\delta^j_l) + \theta_{ij}(-q)^{2(6-i-j)}(\delta^i_l\delta^j_m - q^{-1}\delta^i_m\delta^j_l). \quad (A.6)$$

Appendix B

$$E_{ijkp}E_{lmnp} = (-q)^{2(i+j+k)-S(i,j,k)-9}\,\hat{O}_{ijk}\delta^i_{[l}\delta^j_m\delta^k_{n]} \quad (B.1)$$

$$E_{ijkp}E_{lmpn} = (-q)^{2(i+j)-S(k,i,j)-4}\,\hat{O}_{ijk}\delta^i_{[l}\delta^j_m\delta^k_{n]} \quad (B.2)$$

$$E_{ijkp}E_{lpmn} = (-q)^{2i-S(i,k,j)-2}\,\hat{O}_{ijk}\delta^i_{[l}\delta^j_m\delta^k_{n]} \quad (B.3)$$

$$E_{ijkp}E_{plmn} = (-q)^{S(k,j,i)+3}\,\hat{O}_{ijk}\delta^i_{[l}\delta^j_m\delta^k_{n]} \quad (B.4)$$

$$E_{ijpk}E_{lmpn} = (-q)^{2(i+j-k)-S(i,j)+3S(i,k)+3S(j,k)-3}\,\hat{O}_{ijk}\delta^i_{[l}\delta^j_m\delta^k_{n]} \quad (B.5)$$

$$E_{ijpk}E_{lpmn} = (-q)^{2(i+j+k)-S(i,j,k)-9}\,\hat{O}_{ijk}\delta^i_{[l}\delta^j_m\delta^k_{n]} \quad (B.6)$$

$$E_{ijkp}E_{lmnp} = (-q)^{2(i-k)+S(i,j)+3S(i,k)+S(j,k)+2}\,\hat{O}_{ijk}\delta^i_{[l}\delta^j_m\delta^k_{n]} \quad (B.7)$$

$$E_{ijpk}E_{plmn} = (-q)^{-2k+S(j,i,k)+8}\,\hat{O}_{ijk}\delta^i_{[l}\delta^j_m\delta^k_{n]} \quad (B.8)$$

$$E_{ipjk}E_{lpmn} = (-q)^{2(i-j-k)+3S(i,j)-S(j,k)+3S(i,k)+7}\,\hat{O}_{ijk}\delta^i_{[l}\delta^j_m\delta^k_{n]} \quad (B.9)$$

$$E_{ipjk}E_{plmn} = (-q)^{-2(j+k)+S(i,k,j)+13}\,\hat{O}_{ijk}\delta^i_{[l}\delta^j_m\delta^k_{n]} \quad (B.10)$$

$$E_{pijk}E_{plmn} = (-q)^{-2(i+j+k)+S(k,j,i)+18}\,\hat{O}_{ijk}\delta^i_{[l}\delta^j_m\delta^k_{n]} \quad (B.11)$$

where $\hat{O}_{ijk}$ is an operation which forces i, j, k to be arranged in the order that a smaller index lies to the left. For example, for $k < i < j$,

$$\hat{O}_{ijk}\delta^i_{[l}\delta^j_m\delta^k_{n]} = \delta^k_{[l}\delta^i_m\delta^j_{n]}.$$

The q-symmetrizer is defined as

$$\delta^i_{[l}\delta^j_m\delta^k_{n]} = \delta^i_l\delta^j_m\delta^k_n - q\delta^i_m\delta^j_l\delta^k_n - q\delta^i_l\delta^j_n\delta^k_m + q^2\delta^i_m\delta^j_n\delta^k_l + q^2\delta^i_n\delta^j_l\delta^k_m - q^3\delta^i_n\delta^j_m\delta^k_l.$$

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